COMMONWEALTH OF PENNSYLVANIA
DEPARTMENT OF PUBLIC INSTRUCTION

COURSE OF STUDY IN MATHEMATICS

YEARS VII--XII

HARRISBURG, PENNSYLVANIA.
J. L. L. KUHN, PRINTER TO THE COMMONWEALTH
1923

COMMONWEALTH OF PENNSYLVANIA

DEPARTMENT OF PUBLIC INSTRUCTION

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SYLLABUS IN MATHEMATICS FOR HIGH SCHOOLS

GENERAL STATEMENT

Algebra and geometry originally college subjects. A study of the mathematical courses and entrance requirements of early American colleges shows that algebra and geometry were originally taught in the colleges, and that later both subjects were dropped down bodily into the high schools and academies, becoming college preparatory subjects. The high schools and academies then taught identically the same courses that the colleges formerly had given, using precisely the same textbooks. The mathematics curriculum for the secondary school thus came to be organized from the standpoint of the adult mind, resulting in a formal, logical classification of subject matter.

Present organization is traditional. Our high school mathematics courses have traditionally been organized so as to keep each subject separate from the others: algebra first, then plane geometry, and so on. This organization delays the pupil's introduction to very significant fields of elementary mathematics, such as numerical trigonometry, the use of logarithms, and other tables, graphical methods, and elementary notions of statistics, and denies him the stimulus and satisfaction that comes with real applications of elementary mathematics to the solution of problems.

The conventional organization, furthermore, carries the study of the technique and content of these isolated subjects to a point of completeness beyond that which can be justified for immature learners.

Recommendations growing out of recent studies. The world-wide movement for the reorganization of mathematics courses and teaching that has been in progress for a number of years has resulted (in our own country) in reports and recommendations by such important bodies as the National Committee on Mathematical Requirements, acting under the auspices of the Mathematical Association of America, and the Commission on the Reorganization of Secondary Education, appointed by the National Education Association. The recommendations of these two bodies agree that it is necessary to give up "logical" arrangement of subject matter, especially for introductory work, and to find instead an organization based upon the successful attack of projects and problems in connection with which the pupils already have both knowledge and potential interest. They

likewise agree in advocating the inclusion of arithmetic, algebra, and geometry in the course of study extending through the ninth school year.

General mathematics. Large numbers of pupils leave the high school by the close of the ninth school year. For these pupils training and instruction in "general" or "composite" mathematics, comprising the fundamental ideas of the various branches of elementary mathematics, are more widely applicable to the experiences of everyday life than are the training and instruction limited to algebra alone. For the pupils who will remain in school and continue the study of mathematics, "general" or "composite" courses provide the best foundation for the study of mathematics in the second and later years.

Fundamental principles. It is agreed that the emphasis should be placed throughout on the immediate values of the mathematics subjects studied—not preparation for the study of more mathematics at a remote date in the future, but the acquisition of knowledge and training that will enable the pupil to understand the relations of quantity and space entering into his daily life now. The course should be so planned that if the pupil is compelled to drop out of school at any time before the end of the term, the subjects studied up to that time shall have given him the most valuable mathematical training that he could receive. So planned, the course will likewise be the most natural and effective introduction to mathematics for those pupils who will continue the study of mathematics.

RESPONSIBILITY OF THE TEACHER

Knowledge of subject. It has been said that mathematics teachers "teach the textbook" to a greater extent than any other group of teachers. A reorganization of materials and methods of instruction can become effective only as the teachers assume responsibility for the choice and organization of the subject matter and presentation. The teacher must know his subject thoroughly, and be able to adapt materials and methods to the varying needs of pupils. No method of presentation, and no particular choice of subject matter, can be permanently valid.

Choice and organization of subject matter. In the absence of textbooks embodying general agreement as to the choice and organization of the materials of instruction recommended for courses in composite mathematics, the teacher must assume responsibility for choosing the needed arithmetic material from any good arithmetic text, the algebra material from any good algebra text, and the geometry material from any good geometry text. To teach high school mathematics so as to give the pupil the best mathematical training he is fitted to

receive—resulting in the broad acquaintance with mathematics as a whole that will give him an understanding of the relations of quantity and space growing out of his every-day environment, and that will enable him to decide wisely whether or not to continue the study of mathematics—calls for real skill in the teacher. To meet this demand is the inspiring task of the mathematics teacher of today.

Professional contacts. No teacher of mathematics can do his work adequately unless he keeps in touch with the movement now going on everywhere in the interest of reform and improvement in the teaching of mathematics. This demands interest and participation in the work of such organizations as the Mathematical Association of America, the Association of Mathematics Teachers of the Middle States and Maryland, and the National Council of Teachers of Mathematics. The thoughtful reading of professional literature, including books and periodicals, is absolutely essential to successful teaching of high school mathematics. In particular, it is assumed that teachers will become familiar with standard practice material and with standard tests in high school mathematics subjects, and that they will employ both types of material in classroom practice.

MATHEMATICS OF THE JUNIOR HIGH SCHOOL

Fundamental principles. In the junior high school, comprising years seven, eight, and nine, the mathematics work should give the pupil as broad an outlook on the whole field of mathematics as he is able to comprehend, to the end that he may try out his capacities and aptitudes and learn whether or not he wishes to go more deeply into the study of mathematical topics and to take up the study of scientific and technical subjects that demand extensive mathematical equipment. In addition, the mathematics work of these years should provide the acquaintance with fundamental notions of elementary mathematics that has come to be regarded as essential to intelligent citizenship in the world of today.

Content. The most authoritative opinions available, as expressed in the report and recommendations of the National Committee on Mathematical Requirements, agree that the mathematics course in the junior high school should comprise arithmetic, the elementary notions of intuitive geometry and of algebra, and numerical trigonometry. The following paragraphs present, in outline, the topics that may appropriately be included under each of these headings.

CONTENT OF JUNIOR HIGH SCHOOL MATHEMATICS

Arithmetic

A. *Practice in the fundamental operations* applied to integers and fractions, common and decimal, should be continued until standard

proficiency is attained. When pupils attain standards, they should be excused from regular drill work.

1 In working with fractions, the emphasis is to be placed on simple fractions: such as ½, ⅓, ⅔, ¼, ¾, ⅕, ⅛, et cetera.

2 The decimal equivalents of fractions most commonly used should be fixed in mind, and the process of reducing any fraction to a decimal should be automatized. Rapid drill in one-step operations should be used for speed and accuracy. Material for this purpose may be of the following type:

⅕ of 60 = ?
.20 of 60 = ?
12 = What decimal part of 60?
12 = What fractional part of 60?

3 The accurate placing of the decimal point in the process of division should be made automatic by fixing the habit of writing the quotient over the dividend, and of making the divisor an integer by multiplying, if necessary, both dividend and divisor by the appropriate power of ten.

4 Simple short cuts in multiplication and division: such as replacing multiplication by 25 by multiplication by 100, and division of this result by 4.

5 In the solution of problems, care is to be taken to pass from the solution of the particular concrete problem to the formulation of a general rule.

The sequence, starting with the problem stated in numbers, through the problems stated in letters, the rule, the formula, and the problem without numbers, may be illustrated by the following series. The differences in degree of difficulty vary from step to step, and care must be taken that the pupils are not unduly hurried in the generalizations.

a The length of a sidewalk is 50 ft., and its width is 3 ft. What is its area?

b The length of a sidewalk is *l* ft. and its width *w;* What is A (the area) in terms of *l* and *w?*

c State the rule for finding the area when length and and width are known.

d. Write the formula for A, when l and w are given. Write the formula for l when A and w are given.

e The length and width of a sidewalk are given—How may I figure the cost of laying it, if the cost of one square yard is given?

6 In general, problems involving long computations should not be done in class time.

B Tables of weights and measures in common use. United States money, avoirdupois weight, dry measure, liquid measure linear measure, square measure, cubic measure, time measure.

C Percentage

1 Fixing the equivalence of meanings of the symbols for fraction, decimal, and per cent. Graphs and diagrams to be used as helps in visualizing these equivalences.

2 Interchanging common fractions and percentages, finding any per cent of a number, finding what per cent one number is of another, finding a number when a certain per cent of it is known; such applications of percentage as come within the pupil's experience.

3 Much material must be provided for quick mental work, of the following type:

50% of $200 = ?$

$\frac{1}{2}$ of $200 = ?$

40 is what per cent of 200?

4 Application of percentage

a Interest

The general method of figuring interest should be taught, and also the use of interest tables. Thrift and interest, studied in connection with savings accounts. Compound interest.

b Profit and loss.

Stress the need of care in choosing the base on which to compute the per cent of prifit or loss. Usually this base is the cost: sometimes the selling price is used.

c Commission.

To be computed on the amount of purchases or sales handled by the agent.

d Discount.

Reasons for "2% off for cash"

Successive discounts.

5 The following sequence illustrates desirable work to be done under the applications of percentage. Care must be taken to recognize the increase in difficulty as soon as generalization is undertaken, and to anticipate these difficulties.

a A house was bought for \$5600, and sold for \$6150. What was the per cent of gain?

b A house was bought for P dollars, and sold for S dollars: What was the per cent of gain?

c State the rule for finding the per cent of gain or loss when an article is sold.

d Write the formula for gain per cent, in terms of P (purchase price) and S (Selling price), when S is greater than P.
Write the formula for a loss per cent.

e How shall I find the gain per cent in a transaction which involves buying goods, paying cartage on them to the store, and then selling them?

D Graphic Representation

Making and interpreting line, bar, and circle graphs and applying them wherever they can be used to advantage: care to be taken in making and interpreting pictorial representations, because of the danger of misrepresenting facts when more than one dimension enters into the graph. Much material for the application of graphical methods will be found in the field of social studies and geography.

Coöperation between the teachers of these classes and the mathematics teacher is urgently recommended.

E Business Practice

1 Keeping accounts: family budgets, personal budgets, family accounts, personal accounts.

2 Arithmetic of the community: insurance
Discussion of the need for insurance.
a Fire insurance
b Insurance of other kinds: plate glass, automobile, hailstorm, et cetera.
c Life and accident insurance.
Problems on insurance to involve only the simple applications.

3 Arithmetic of civic life: taxes.
a City taxes.
Sources of expense in local governments, and ways of levying taxes.
b National revenue.
Sources of expense in national governments, and ways of raising revenue: customs and duties, income tax, inheritance tax, luxury tax, et cetera.

4 Arithmetic of banking.
a Different kinds of banks.
b Savings account and checking account: deposit slips for each.

c Writing and endorsing a check.
d Making out withdrawal slip for savings account.
e Borrowing money: promissory note.
f Transmitting money to distant places.

5 Arithmetic of investment.
a Buying and selling real estate.
b Stocks and bonds: fix clearly the difference between these two as investments. Use market reports of newspapers for problem material. Is a bond promising 10% return always better than one promising 6%? Importance of dealing with reliable agents: consulting with the local banker.
c Commission and brokerage applied to real estate and stocks and bonds.

Intuitive Geometry

This subject should acquaint the pupil in a simple and interesting way with the most important geometric forms and their applications, through directed observation and experiment.

A Simple Geometrical Figures. Familiarity with properties of equilateral triangle, 30°—60° right triangle, isosceles right triangle, circle, square, regular hexagon: symmetry, axial and central; knowledge of the facts concerning the sum of the angles in a triangle; the pythagorean theorem; simple solids, such as a cube, pyramid, cone, prism, sphere. In connection with this work, accustom pupils from the beginning to using "circle" to mean the closed curve, and "polygon" to mean the closed broken line. "Area of the circle" and "area of the polygon" are to be taken to mean the areas enclosed by the lines.

B Simple Geometrical Drawing. Use of T-square, triangles, protractor and compasses: constructing perpendicular bisectors of lines, bisectors of angles, parallel lines; constructing triangles from given data; regular polygons, simple designs for ornament, et cetera. Use of squared paper.

C Direct Measurements. Use of linear scales and protractor. Appreciation of the fact that these measurements are approximations, and development of judgment in the use of such data in computation: particular attention to be given to the number of figures to be retained in computations with approximate data.

D Indirect Measurements. Making simple drawings to scale from actual measurements made by the pupil, and using them to get data not secured directly. Such applications to measuring heights and distances as are given in the Boy Scouts Handbook.

E Informal treatment of the idea of similarity. Drawing to scale: plans, working drawings, maps; use of squared paper; simple applications of proportion.

F Mensuration. Area of square, rectangle, parallelogram, triangle, and trapezoid; length and area of circle; surfaces and volumes of solids of corresponding importance. These facts to be established in an experimental way, so far as possible by the pupil's own activity. In the mensuration of the circle, use the value $\frac{22}{7}$ for π, avoiding the use of 3.1416, except in the rare cases where it is warranted by the exactness of the measurements.

G Geometry of Appreciation. Geometrical forms in nature and art; in flowers, fruit, leaves, and animal forms; in architecture, manufacture, and industry.

The pupil's active participation in discovering and describing the geometrical forms here mentioned is essential. The work in intuitive geometry has general educational value in every day affairs, and serves as an introduction to demonstrative geometry. Along with the acquisition of fundamental ideas concerning the size, shape, and position of geometric forms in the plane and in space, the pupil should begin to make inferences and draw valid conclusions from facts discovered experimentally.

Algebra

A The formula. The construction of formulas by the pupil, as the outcome of work in mensuration. The following may serve as types:

$D = r. t.$ (Distance in terms of rate and time)
$A = 1/2\ b. h.$ (Area of triangle)
$P = 2\ l + 2\ h$ (Perimeter of rectangle)
$V = b. w. h.$ (Volume of rectangular solid)
$C = \pi d$. (Circle)

At a later stage, the pupils will be encouraged to bring to class formulas they have encountered in their reading, and to evaluate and explain them.

Care should be taken that the material used is not too far separated from the pupil's own experience.

B Graphic representation. Statistical data from the field of geography and the social studies supply the material for the first use of graphs.

Probably the first point at which the graph will be appreciated by the pupil in the study of algebra will be in connection with the study of the number-pairs that satisfy an equation like $2x - 3y = 46$, and their determination of the points that constitute the graph of the equation.

The graph is to be used throughout the study of high school mathematics and is not to be taught as a separate topic.

Whenever a diagram can be made to help in understanding a problem or discussion, the pupil should be encouraged to make use of it, until the making of such graphs or diagrams becomes a fixed habit.

C Positive and Negative Numbers. Their meaning and use as expressing both magnitude and one of two opposite directions or senses.

Their graphic representation in connection with the representation of real numbers by the points on a line, emphasizing the fact that of two given numbers, that one is the greater whose point-representation lies farther to the right on the axis of real numbers.

The fundamental operations applied to them.

Avoid any attempt to prove the laws of signs for the multiplication of signed numbers: these laws to be given as definitions.

D The Equation. The introductory problems leading to the use of the equation must be very simple so that stress can be laid upon the mastery of the brief and systematic treatment of numerical relations that will later enable the pupil to solve problems otherwise beyond his power.

At first the equation will involve only positive numbers.

The negative numbers that appear later should be treated in a very simple manner, by adding an amount sufficient to make up the amount subtracted. Thus, in the problem: "The sum of two numbers is 94, and their difference is 38: find the numbers."

$$(94-x) - x = 38$$
$$94-2x = 38$$

The $-2x$ is disposed of by adding $2x$ to both sides, making the equation read

$$94 = 38 + 2x,$$

and the 38 on the right side disappears upon substracting 38 from both sides, getting

$$56 = 2x,$$
$$\text{whence } x = 28$$

It is well to accustom the pupil early in his study of algebra to look upon the form $56 = 2x$ as being just as good as the form $2x = 56$.

Throughout the course, stress will be laid upon the solution of problems, involving the translation of the verbal statement into algebraic language.

Equations in two variables will be restricted to sets of linear equations at first. In a first course, elimination by addition and subtraction to be the only method employed: solution to be checked by graphs. The solution of the quadratic equation to be restricted to the "pure" quadratics in one unknown.

A simple treatment of proportion, including various applications of ratio and proportion to the problems of ordinary experience. Use the fractional notation $\frac{b}{a} = \frac{d}{c}$ discarding the archaic "dot" notation. Discard the terms "antecedent" and "consequent," using "numerator" and "denominator" instead.

Algebraic Technique

A. *The Four Fundamental Operations.* Their treatment should include the representation of numerical relations by means of algebraic symbols, and the translation of symbolic expressions into words.

Multiplication and division should rarely involve anything beyond monomial and binomial multipliers, divisors, and quotients. Complications never met in the practical applications of algebra (as, for instance, nests of parentheses) should be avoided.

The treatment of literal equations should be restricted to the material necessary for attaining facility in manipulating formulas.

The solutions of problems should involve the verification of the results obtained. The feeling of self-confidence and certainty that his work is correct is a valuable outcome from the pupil's study of mathematics.

B *Factoring. The only cases that need to be treated are:*

1 Monomial factors, as in $ax + ay + az$.
2 The difference of two squares.
3 The square of a binomial.

C *Fractions.* The four fundamental operations should be applied to simple cases, constantly through the course, avoiding complicated forms that are never met with in practice.

Changing fractions to equivalent fractions having different denominators should be made to depend on the pupil's recognition of the need of multiplying the old numerator by the same factor that the old denominator was multiplied by in obtaining the new denominator. Thus, $\frac{3a}{4b} = \frac{?}{20bc}$ will be completed by answering the questions "What was the denominator 4b multiplied by to make the new denominator 20bc?" and "What, then, must be done to the old numerator 3a?"

"Clearing an equation of fractions" should be rationalized for the pupil, so that he knows that the process involves multiplying every term by a multiplier that will cause the fractions to disappear. Separate treatment of greatest common divisor, and lowest com-

mon denominator, isolated from the treatment of equations involving fractions, is inadvisable.

When such an equation as

$$\frac{3x-4}{5}+\frac{7x-2}{10}=\frac{9x-1}{8}$$

is to be solved, call first for the method of getting rid of the largest denominator. "By multiplying through by 10."—This gives the result

$$\frac{\overset{2}{\cancel{10}}(3x-4)}{\cancel{5}}+\frac{\cancel{10}(7x-2)}{\cancel{10}}=\frac{\overset{5}{\cancel{10}}(9x-1)}{\underset{4}{\cancel{8}}}$$

leaving a denominator 4, which may be made to disappear by multiplying through by 4, giving

$$24x-32+28x-8=45x-5,$$

from which

$$52x-40=45x-5$$

and

$$x=5$$

By the time the majority of the class has acquired facility in this process of solution, some member is fairly sure to suggest replacing the separate multiplications by 10 and 4, by a single multiplication by 40.

Then the class may be set the task of determining the single multipliers that will get rid of all the denominators at one time, in similar equations.

Complex fractions should be restricted to such as are not more difficult than

$$\frac{\frac{a}{b}+\frac{c}{d}}{\frac{m}{n}-\frac{p}{q}}$$

D Exponents and Radicals. The treatment of these topics should be confined to the simplest material needed for the treatment of formulas.

Care should be taken to make it clear that the symbol $\sqrt{a}$ (a representing a positive number) means only the positive square root, and that $\sqrt[n]{a}$ means only the principal *n*th root. It is therefore improper to write $x=\sqrt{3}$ as the complete solution of $x^2-3=0$, of which the result should be written $x=\pm\sqrt{3}$.

In connection with the work with the pythagorean theorem, it will be necessary to teach a process for finding the square roots of numbers. For this purpose, the following process is recommended:

Required, to find the value of $\sqrt{5486}$.

The pupil knows that $70^2 = 4900$

and that $80^2 = 6400$

and consequently that $\sqrt{5486}$ lies between 70 and 80, and is nearer 70 than 80. When required to estimate its value, he may say 72. The test of 72 as the square root is made by dividing it into the given number. The result, 76.2, indicates that 72 is not the true square root, and further that the value of the square root is larger than 72, and smaller than 76.2. The mean value of these two numbers, 74.1, may then be taken as the required value of $\sqrt{5486}$. If a greater degree of accuracy is required, the process of checking by division, and correcting by taking the mean value, may be repeated. The advantage of this process is that it compels the pupils to form correct conceptions of the meaning of square root, and that its simplicity ensures retention. At a later time, the traditional method based on the expansion of $(a + b)^2$ may be taught, if considered desirable.

Numerical Trigonometry

A *Definitions of sine, cosine and tangent,* arrived at in an experimental way, through measurements and tabulations made by the pupil. Constructing angles when values of the functions are given.

B *The use of the functions in solving simple problems in heights and distances.* In this connection it is highly important that the pupils themselves make the measurements needed—that field-work of a very simple kind be required. Instruments constructed by the class are better for this purpose than surveyors' instruments. Measurements and constructions with steel tape; heights by "shadow" method and by measurements of distance and angle of elevation; areas of irregular plots of ground.

C *The construction of very simple tables of the functions by the pupils,* and the use of 3 or 4 place printed tables of natural values of the functions.

In the work done on this topic, attention will be concentrated upon, and limited to, the simplest fundamental notions.

Problems

Must be real to the pupil. Throughout the course in high school mathematics, the solution of problems must be given major emphasis. So far as the pupil's maturity and knowledge of science and industry permit, "practical problems" should be freely introduced in high school mathematics teaching. Care must be taken to ensure that

these problems are real to the pupil, that they connect with his ordinary thoughts, and lie within the world of his interest and experience.

Computation. In the solution of problems, opportunity will continually be presented for arithmetical and computational work. The notion that algebra is an extension of arithmetic should be emphasized both in numerical work and in explaining algebraic principles. Computations and the solutions of problems should habitually be checked. The use of approximate data in computation presents opportunities to stress the need for exercising common sense and judgment especially with regard to the number of significant figures retained in the results of computations. Pupils should gradually be accustomed to the use of such tables as squares and square roots, cubes and cube roots, trigonometric functions, interest, and the like, to facilitate computation.

Algebra Topics to be Omitted

The following topics should be omitted from a first course in high school mathematics:

Drill in algebraic technique designed merely to secure facility and skill in manipulation, apart from the acquisition of power to attack significant problems.
Cases in factoring other than those listed.
Highest common factor and lowest common multiple as separate topics.
The theorems on proportion relating to alternation, inversion, composition, and division.
Literal equations, except such as appear in connection with work on formulas.
Radicals, except as indicated in section D, page 13.
Square roots of polynomials.
Cube root.
Theory of exponents.
Simultaneous equations in more than two unknowns.
Pairs of simultaneous quadratic equations.
The theory of quadratics (remainder and factor theorems, etc.)
All equations of degree higher than the second.
The binomial theorem.
Arithmetic and geometric progressions.
Theory of imaginary and complex numbers.
Radical equations, except such as arise in dealing with elementary formulas.

THE COURSE OF STUDY IN MATHEMATICS FOR THE JUNIOR HIGH SCHOOL

The topics to be included in the Course of Study for the Junior High School may be arranged in the following manner:

Seventh year: Arithmetic, particularly as applied to the home, to industry, and to the other subjects

	in the school curriculum. Intuitive geometry.
Eighth year:	Algebra; arithmetic, particularly its social and commercial applications. Such geometry as grows naturally out of discussions of the size and shape of figures.
Ninth year:	Algebra; numerical trigonometry; where possible, and introduction to demonstrative geometry, with the aim limited to learning the meaning of "demonstration."

NINTH YEAR MATHEMATICS IN THE FOUR-YEAR HIGH SCHOOL

Purpose and content. In four-year high schools, the ninth year's work in mathematics should provide the pupil with as broad a foundation of mathematical training as possible. In particular, the course should include algebra, numerical trigonometry, and geometry, with at least an indication of the nature of a geometric demonstration.

Time allotment. In such schools, it is recommended that about two-thirds of the ninth year be devoted to algebra and numerical trigonometry, and about one-third to geometry. The outline of subject matter for the junior high school mathematics will be of use in connection with the type of school here discussed, provided proper allowance is made for the greater maturity of the pupils, and for the fact that the mensuration work done in the seventh and eighth years of the elementary school has furnished a considerable body of geometric training.

Guidance in electing mathematics. Provision must be made through the guidance program of the school, or otherwise, to ensure that all pupils are properly informed of the vital importance of mathematics in many lines of endeavor in adult life, and to see to it that pupils are properly advised when they decide whether or not to continue the study of mathematics beyond the ninth year.

Transition from algebra to general mathematics. Algebra will continue for some time to be the mathematics taught in the ninth year in many high schools. This is as it should be, in view of the fact that radical departure from customary practice without preparation would be unwise. However, in view of the advantage of the course in mathematics for the ninth year comprising the fundamental notions of algebra, intuitive geometry and numerical trigonometry, it is confidently expected that gradually ninth year algebra will be replaced by a year of composite mathematics. There will be a period of transition, in which the material of instruction will be organized by teachers, eliminating a considerable

body of algebra material that has been customarily given, and bringing in new material from the geometry side. The final result of this process should be a year's work involving the fundamental notions of the elementary branches of mathematics that will contribute most largely to the pupil's training for intelligent citizenship, as well as furnish the best foundation for further work in mathematics.

COURSE IN ALGEBRA COVERING NINTH YEAR

Ninth Year Algebra. The following outline is designed to serve the needs of schools in which the ninth year is devoted to the study of algebra.

The Formula. Material leading to the formula will have been met in the pupil's work in arithmetic, in connection with such topics as interest, and mensuration. This experience with the formula should be reviewed, and extended.

Special care should be taken to see that the pupils understand the algebraic language into which the verbal statements are translated.

The dependence of one quantity upon another, as expressed by the formula, should be clearly understood. Such questions as the following are appropriate:

Given the formula $V=1/3\pi r^2h$. (a) If V is to be kept constant how must h change when r increases? (b) If r is kept fixed, how will V change when h is doubled?

In the science class, there will be occasion to study such formulas as:

Ohm's Law: $I = \frac{E}{R}$

Law of machines: $P \times P\,d = W \times W\,d$

Thermometer: $C = \frac{5}{9}(F-32)$

Such formulas as these should be taken up in the algebra classroom as one phase of coöperation between the teachers of science and mathematics.

Graphic Representation. This should not be considered an isolated topic, but should be used throughout the year's work, whenever helpful, as an illustrative and interpretative instrument.

Value coördination may be effected by coördination between the mathematics teacher and the science teacher, in making graphs for such topics as the following:

Production of coal, petroleum, etc., in the United States.

Distribution of elements in earth's crust.

Range of wave-lengths of heat, sound, wireless, etc.
Cost of gas used in various appliances.
Food values of various substances.
Hours of light and darkness through the year.
Health statistics.
Temperature graphs.

Positive and Negative Numbers. Their meaning and use as expressing both magnitude and one of two opposite directions or senses.

Their graphic representation in connection with the representation of real numbers by the points on a line, emphasizing the fact that of two given numbers, that one is the greater whose point-representation lies farther to the right on the axis of real numbers.

The fundamental operations applied to them. No attempt should be made to prove the laws of signs for the multiplication and division of signed numbers; these laws should be given as definitions.

The Equation. (a) Linear equations in one unknown—setting up such equations as translations of verbally given problems, and solving them. There should be much drill at first on very simple types of equations.

The use of "transpose" as a technical term should be postponed until late in the year, using at first the phrase "add to both sides" or "subtract from both sides." The habit of combining terms in both members of the equation should be established as the first step in solving an equation.

Thus the equation:

$$2x + 1 - 3x + 7 = x + 9 - 5x$$

should first be simplified to

$$-x+8=-4x+9$$

and then, by adding 4x to both sides, and subtracting 8 from both sides, we have

$$3x = 1$$

(b) Simple cases of quadratic equations that arise in solving problems and in handling formulas.

(c) Sets of equations in two unknowns, limited to pairs of linear equations.

(d) Applications of ratio and proportion to simple cases of similarity and other problems of ordinary life. The proportion should always be written as an equation between two fractions, and in solving the proportion, it should be treated as an equation. The terms "means," "extremes," "antecedent," "consequent," should be discarded in favor of "numerator" and denominator."

Algebraic Technique. The Four Fundamental Operations. Their connection with the processes of arithmetic should be made clear.

Multiplication and division should rarely involve multipliers or divisors of more than two terms.

"Nests" of parentheses should not be treated, because of their rare occurrence in practical applications.

Literal equations should be treated only to the extent necessary for manipulating formulas.

The habit of verifying solutions should be established.

Factoring. The only cases that need to be treated are:

1. Monomial factors.
2. The difference of two squares.
3. The square of a binomial.

Skill acquired in factoring other cases—as the sum of two cubes, for instance—is wasted because no opportunity is ever presented for its use in applications of mathematics to real situations.

Fractions. The four fundamental operations should be applied only to simple cases, throughout the course, avoiding complicated forms that are never met with in practice.

The use of "cancel" as a technical term should be postponed until late in the year, using instead the phrase "divide numerator and denominator by."

Errors of the type $\frac{ax}{ay + z} = \frac{x}{y + z}$ should be forestalled by insistence upon division of both terms of the fraction in this simplification.

Separate treatment of "least common denominator" and "highest common divisor" should not be given. "Clearing an equation of fractions" should be rationalized as indicated on page 13.

Complex fractions should be restricted to such as are not more difficult than

$$\frac{\frac{a}{b} + \frac{c}{d}}{\frac{m}{n} - \frac{p}{q}}$$

Exponents and Radicals. The work done on exponents and radicals should be confined to the simplest material required for the treatment of formulas.

Proofs of the laws for positive integral exponents should be included. Care should be taken in this connection to ensure understanding of the facts dealt with, by illustrations from the arithmetic

side. Thus appreciation of the truth of $2^2 \times 2^3 = 2^5$, by translating into $4 \times 8 = 32$, should precede learning $x^a \times x^b = x^{a+b}$.

The meaning and use of fractional and negative exponents should be considered in connection with handling formulas. With some classes, it will be possible to include an elementary discussion of logarithms and the slide rule.

The consideration of radicals should be confined to transformations of the types:

$$\sqrt{a^2b} = a\sqrt{b} \text{ and } \sqrt{a/b} = 1/b\sqrt{ab}.$$

A process for finding the square root of a number should be taught (see page 14), but time should never be given to extracting roots of algebraic polynomials.

Problems. Most of the emphasis now frequently placed on formal exercises should be transferred to the solution of problems. Problems should be "practical" so far as the maturity of the pupil permits.

Problems should always be "real" to the pupil, should connect with his ordinary thought, and be within the world of his interests and experience.

A conscious effect should be made, in the selection of problems, to correlate the work in mathematics with the other courses in the curriculum, particularly with the courses in science.

Numerical Trigonometry. In view of the fact that this subject has been recommended for inclusion in the College Entrance Examination Board Examination in Elementary Algebra (Part I), Algebra to Quadratics, it probably will be considered desirable to include numerical trigonometry with algebra in all schools devoting the whole ninth year to algebra.

An outline will be found on page 14.

TENTH YEAR MATHEMATICS

Election and Guidance. It is assumed that the ninth year's work in mathematics is the final year's work in required mathematics. Later work in this subject will be elective, and elections will be made under the guidance of competent teacher-advisers. This assumption of adjustment to the individual needs and aptitudes of pupils is increasingly justified as proper recognition is given to the importance of guidance activities in the high school.

Demonstrative geometry. On the basis of the work done in preceding years, whether in junior high school or in the elementary school and ninth year in the four-year high school, the pupil who elects to continue the study of mathematics will be ready to take up the study of demonstrative geometry. In some schools the training in intuitive geometry already secured will enable the pupil to cover

the work in plane and solid geometry in one year. In this case the greater part of this year will be devoted to plane geometry, and the lesser part (say, about one-third) to solid geometry.

The work in demonstrative geometry should always be preceded by work in intuitive geometry, such as is outlined on page 9 above. If the pupils beginning demonstrative geometry have not already had this type of work, a short time (say, three weeks) should be devoted to such introductory work before proceeding to the strictly demonstrative type of work.

Assumptions and Theorems for Informal Treatment. This list contains propositions which may be assumed without proof (postulates), and theorems which it is permissible to treat informally. Some of these propositions will appear as definitions in certain methods of treatment. Moreover, teachers should feel free to require formal proofs of some, if they desire to do so. The precise wording given is not essential, nor is the order in which the propositions are here listed:

1 Through two distinct points it is possible to draw one straight line, and only one.
2 A line segment may be produced to any desired length.
3 The shortest path between two points is the line segment joining them.
4 One and only one perpendicular can be drawn through a given point to a given straight line.
5 The shortest distance from a point to a line is the perpendicular distance from the point to the line.
6 From a given center and with a given radius one and only one circle can be described in a plane.
7 A straight line intersects a circle in at most two points.
8 Any figure may be moved from one place to another without changing its shape or size.
9 All right angles are equal.
10 If the sum of two adjacent angles equals a straight angle, their exterior sides form a straight line.
11 Equal angles have equal complements and equal supplements.
12 Vertical angles are equal.
13 Two lines perpendicular to the same line are parallel.
14 Through a given point not on a given straight line one straight line, and only one, can be drawn parallel to the given line.
15 Two lines parallel to the same line are parallel to each other.

16 The area of a rectangle is equal to its base times its altitude.

Fundamental Theorems. The following list of theorems in intended to include those of sufficient importance to require demonstration:

1 Two triangles are congruent if*
   a Two sides and the included angle of one are equal, respectively, to two sides and the included angle of the other;
   b Two angles and a side of one are equal, respectively, to two angles and the corresponding side of the other;
   c The three sides of one are equal, respectively, to the three sides of the other.

2 Two right triangles are congruent if the hypotenuse and one other side of one are equal, respectively, to the hypotenuse and another side of the other.

3 If two sides of a triangle are equal the angles opposite these sides are equal; and conversely.**

4 The locus of a point (in a plane) equidistant from two given points is the perpendicular bisector of the line segment joining them.

5 The locus of a point equidistant from two given intersecting lines is the pair of lines bisecting the angles formed by these lines.

6 When a transversal cuts two parallel lines, the alternate interior angles are equal; and conversely.

7 The sum of the angles of a triangle is two right angles.

8 A parallelogram is divided into congruent triangles by either diagonal.

9 Any (convex) quadrilateral is a parallelogram, (a) if the opposite sides are equal; (b) if two sides are equal and parallel.

10 If a series of parallel lines cut off equal segments on one transversal, they cut off equal segments on any transversal.

* Teachers should feel free to separate this theorem into three distinct theorems and to use other phraseology for any such proposition. For example in 1, "Two triangles are equal if.." "A triangle is determined by ....", etc. Similarly, in 2, the statement might read: "Two right triangles are congruent if, besides the right angles, any two parts (not both angles) in the one are equal to corresponding parts of the other."

** It should be understood that the converse of a theorem need not be treated in connection with the theorem itself, it being sometimes better to treat it later. Furthermore, a converse may occasionally be accepted as true in an elementary course, if the necessity for proof is made clear. The proof to be given later.

11 a The area of a parallelogram is equal to the base times the altitude.

b The area of a triangle is equal to one half the base times the altitude.

c The area of a trapezoid is equal to half the sum of its bases times its altitude.

d The area of a regular polygon is equal to half the product of its apothem and perimeter.

12 a If a straight line is drawn through two sides of a triangle parallel to the third side, it divides these sides proportionally.

b If a line divides two sides of a triangle proportionally, it is parallel to the third side. (Proof for commensurable case only.)

c The segments cut off on two transversals by a series of parallels are proportional.

13 Two triangles are similar if

a They have three angles of one equal, respectively, to three angles of the other;

b They have an angle of one equal to an angle of the other and the including sides are proportional;

c Their sides are respectively proportional.

14 If two chords intersect in a circle, the product of the segments of one is equal to the product of the segments of the other.

15 The perimeters of two similar polygons have the same ratio as any two corresponding sides.

16 Polygons are similar, if they can be decomposed into the same number of triangles, similar each to each and similarly placed; and conversely.

17 The bisector of an (interior or exterior) angle of a triangle divides the opposite side (produced if necessary) into segments proportional to the adjacent sides.

18 The areas of two similar triangles (or polygons) are to each other as the squares of any two corresponding sides.

19 In any right triangle the perpendicular from the vertex of the right angle on the hypotenuse divides the triangle into two triangles each similar to the given triangle.

20 In a right triangle the square on the hypotenuse is equal to the sum of the squares on the other two sides.

21 In the same circle or in equal circles, if two arcs are equal, their central angles are equal; and conversely.

22 In any circle two angles at the center are proportional to their intercepted arcs. (Proof for commensurable case only)

23 In the same circle or in equal circles, if two arcs are equal their chords are equal; and conversely.

24 a A diameter perpendicular to a chord bisects the chord and the arcs of the chord.

b A diameter which bisects a chord (that is not a diameter) is perpendicular to it.

25 The tangent to a circle at a given point is perpendicular to the radius at that point; and conversely.

26 In the same circle or in equal circles, equal chords are equally distant from the center; and conversely.

27 An angle inscribed in a circle is equal to half the central angle having the same arc.

28 Angles inscribed in the same segment are equal.

29 If a circle is divided into equal arcs, the chords of these arcs form a regular inscribed polygon and tangents to the points of division form a regular circumscribed polygon.

30 The circumference of a circle is equal to $2 \pi r$ (informal proof only).

31 *The area of a circle is equal to πr^2 (informal proof only)

The treatment of the mensuration of the circle should be based on related theorems concerning regular polygons, but it should be informal as to the limiting processes involved. The aim should be an understanding of the concepts involved, so far as the capacity of the pupil permits. It is recommended that the theorems concerning the length of side of a polygon of 2n sides, in terms of the side of the polygon having n sides, for the inscribed and circumscribed polygons, be omitted. The value of π should be given to four or five places, with the statement that its value to a greater degree of accuracy is found by methods of advanced mathematics. For computation purposes the value $\frac{22}{7}$ should be used.

FUNDAMENTAL CONSTRUCTIONS

1 Bisect a line segment and draw the perpendicular bisector.

*The total number of theorems given in this list when separated (as will be found advantageous in teaching) including the converses indicated, is 52.

2 Bisect an angle.
3 Draw a perpendicular to a given line through a given point.
4 Construct an angle equal to a given angle.
5 Through a given point draw a straight line parallel to a given straight line.
6 Construct a triangle, given
  a Three sides
  b Two sides and the included angle
  c Two angles and the included side
7 Divide a line segment into parts proportional to given segments.
8 Given an arc of a circle, find its center.
9 Circumscribe a circle about a triangle.
10 Inscribe a circle in a triangle.
11 Construct a tangent to a circle through a given point.
12 Construct the fourth proportional to three given line segments.
13 Construct the mean proportional between two given line segments.
14 Construct a triangle (polygon) similar to a given triangle (polygon).
15 Construct a triangle equal to a given polygon.
16 Inscribe a square in a circle.
17 Inscribe a regular hexagon in a circle.

Subsidiary Propositions. The following list is intended to include material from which additional theorems, corollaries, originals, and exercises may be selected:

1 When two lines are cut by a transversal, if the corresponding angles are equal, or if the interior angles on the same side of the transversal are supplementary, the lines are parallel.
2 When a transversal cuts two parallel lines, the corresponding angles are equal, and the interior angles on the same side of the transversal are supplementary.
3 A line perpendicular to one of two parallels is perpendicular to the other also.
4 If two angles have their sides respectively parallel or respectively perpendicular to each other, they are either equal or supplementary.
5 Any exterior angle of a triangle is equal to the sum of the two opposite interior angles.
6 The sum of the angles of a convex polygon of n sides is 2(n—2) right angles

7 In any parallelogram
  a the opposite sides are equal;
  b the opposite angles are equal;
  c the diagonals bisect each other.

8 Any (convex) quadrilateral is a parallelogram, if
  a the opposite angles are equal;
  b the diagonals bisect each other.

9 The medians of a triangle intersect in a point which is two-thirds of the distance from the vertex to the mid-point of the opposite side.

10 The altitudes of a triangle meet in a point.

11 The perpendicular bisectors of the sides of a triangle meet in a point.

12 The bisectors of the angles of a triangle meet in a point.

13 The tangents to a circle from an external point are equal.

14 *a If two sides of a triangle are unequal, the greater side has the greater angle opposite it; and conversely.
  b If two sides of one triangle are equal respectively to two sides of another triangle, but the included angle of the first is greater than the included angle of the second, then the third side of the first is greater than the third side of the second; and conversely.
  c If two chords are unequal; the greater is at the less distance from the center, and conversely.
  d The greater of two minor arcs has the greater chord and conversely.

15 An angle inscribed in a semi-circle is a right angle.

16 Parallel lines, tangent to or cutting a circle, intercept equal arcs on the circle.

17 An angle formed by a tangent and a chord of a circle is measured by half the intercepted arc.

18 An angle formed by two intersecting chords is measured by half the sum of the intercepted arcs.

19 An angle formed by two secants, or by two tangents, to a circle, is measured by half the difference between the intercepted arcs.

20 If from a point without a circle a secant and a tangent are drawn, the tangent is the mean proportional between the whole secant and its external segment.

21 Parallelograms, or triangles, of equal bases and altitudes are equal.

*Such inequality theorems as these are of great importance in developing the notion of dependence or functionality in geometry. The fact that they are placed in this "Subsidiary List of Propositions" should not imply that they are considered of less educational value than those in List II. They are placed here because they are not "fundamental" in the same sense that the theorems of List II are fundamental.

22 The perimeters of two regular polygons of the same number of sides, are to each other as their radii and also as their apothems.

Solid Geometry A: combined with plane geometry in one year. For classes desiring to cover plane and solid geometry in one year, as indicated on page 21, the study of solid geometry should include the following:

An informal treatment of the fundamental theorems dealing with the relations of lines and planes in space, based on the pupil's experience, and familiarity with the substance of the usual definitions and theorems.

An informal treatment of locus problems, emphasizing the acquisition of power to visualize, describe and represent the figures dealt with, and putting relatively little emphasis upon formal proofs.

Problems of measurement and calculation, stated in words or in terms of a figure, form the most important topic in the work on solid geometry. The pupil should be familiar with the mensuration formulas of plane geometry, with the use of the sine and tangent functions in solving right triangles, and in the course of the work should become familiar with the solid geometry formulas listed below.

Similarity of solids should be carried far enough to make the pupil familiar with the facts that the volumes of similar solids are proportional to the cubes, and surfaces to the squares, of corresponding dimensions.

Abilities to be acquired. Throughout the course, stress should be laid upon gaining ability to visualize the space relations of the figures dealt with, to describe them accurately in words, to represent them clearly by drawings, free-hand or made with instruments. The pupil should become able to use intelligently the terminology of the subject, including such terms as polyhedron, regular polyhedron, prism, cylinder, pyramid, cone, sphere; right, regular and oblique, applied to solids; vertex, diagonal, face diagonal, lateral edge, altitude, slant height; zone of a sphere, lune, great circle, small circle; similar solids.

Fundamental theorem in mensuration of volumes. Mensuration of volumes may be made to depend upon an informal treatment of the theorem:

If two solids are included between two parallel planes, and are such that sections made by any plane parallel to the first two are equal in area, then the solids are equal in volume.

Mensuration formulas. The following list of mensuration formulas should be included:

Prism: lateral surface, volume,

Right circular cylinder: lateral surface, total surface, volume.

Regular pyramid and right circular cone and their frustums: lateral surface and volume.

Sphere: Surface, area of zone, area of lune of n degrees, volume.

ELEVENTH YEAR MATHEMATICS

Subject matter. The mathematics work of the eleventh year may be divided between advanced algebra and solid geometry, when the latter subject is not taken in the tenth year. The order in which these two topics is given is unimportant. When solid geometry is taken up in connection with plane geometry in the tenth year, the eleventh year will be divided between advanced algebra and trigonometry.

For some groups of pupils, a course in shop mathematics may wisely be given in the eleventh year in place of the algebra-solid geometry or algebra-trigonometry just indicated. Such a course is referred to briefly on page 37.

Solid Geometry B: Covering One-half Year. For classes devoting a half year to solid geometry, the following list of propositions is recommended. The division into "fundamental" and "subsidiary" theorems will again call attention to the need for varying the emphasis given different portions of this material of instruction.

In the following list the precise wording and the sequence are not considered:

I. FUNDAMENTAL THEOREMS

1. If two planes meet, they intersect in a straight line.
2. If a line is perpendicular to each of two intersecting lines at their point of intersection it is perpendicular to the plane of the two lines.
3. Every perpendicular to a given line at a given point lies in a plane perpendicular to the given line at the given point.
4. Through a given point (internal or external) there can pass one and only one perpendicular to a plane.
5. Two lines perpendicular to the same plane are parallel.
6. If two lines are parallel, every plane containing one of the lines and only one is parallel to the other.
7. Two planes perpendicular to the same line are parallel.
8. If two parallel planes are cut by a third plane, the lines of intersection are parallel.

9. If two angles not in the same plane have their sides respectively parallel in the same sense, they are equal and their planes are parallel.
10. If two planes are perpendicular to each other, a line drawn in one of them perpendicular to their intersection is perpendicular to the other.
11. If a line is perpendicular to a given plane, every plane which contains this line is perpendicular to the given plane.
12. If two intersecting planes are each perpendicular to a third plane, their intersection is also perpendicular to that plane.
13. The sections of a prism made by parrallel planes cutting all the lateral edges are congruent polygons.
14. An oblique prism is equal to a right prism whose base is equal to a right section of the oblique prism and whose altitude is equal to a lateral edge of the oblique prism.
15. The opposite faces of a parallelopiped are congruent.
16. The plane passed through two diagonally opposite edges of a parallelopiped divides the parallelopiped into two equal triangular prisms.
17. If a pyramid or a cone is cut by a plane parallel to the base:
    (a) The lateral edges and the altitude are divided proportionally;
    (b) The section is a figure similar to the base;
    (c) The area of the section is to the area of the base as the square of its distance from the vertex is to the square of the altitude of the pyramid or cone.
18. Two triangular pyramids having equal bases and equal altitudes are equal.
19. All points on a circle of a sphere are equidistant from either pole of the circle.
20. On any sphere a point which is at a quadrant's distance from each of two other points not the extremities of a diameter is a pole of the great circle passing through these two points.
21. If a plane is perpendicular to a radius at its extremity on a sphere, it is tangent to the sphere.
22. A sphere can be inscribed in or circumscribed about any tetrahedron.
23. If one spherical triangle is the polar of another, then reciprocally the second is the polar triangle of the first.

24. In two polar triangles each angle of either is the supplement of the opposite side of the other.
25. Two symmetric spherical triangles are equal.

II. FUNDAMENTAL PROPOSITIONS IN MENSURATION

26. The lateral area of a prism or a circular cylinder is equal to the product of a lateral edge or element respectively, by the perimeter of a right section.
27. The volume of a prism (including any parallelopiped) or of a circular cylinder is equal to the product of its base by its altitude.
28. The lateral area of a regular pyramid or a right circular cone is equal to half the product of its slant height by the perimeter of its base.
29. The volume of a pyramid or a cone is equal to one-third the product of its base by its altitude.
30. The area of a sphere.
31. The area of a spherical polygon.
32. The volume of a sphere.

III. SUBSIDIARY THEOREMS

33. If from an external point a perpendicular and obliques are drawn to a plane, (a) the perpendicular is shorter than any oblique; (b) obliques meeting the plane at equal distances from the foot of the perpendicular are equal; (c) of two obliques meeting the plane at unequal distances from the foot of the perpendicular, the more remote is the longer.
34. If two lines are cut by three parallel planes, their corresponding segments are proportional.
35. Between two lines not in the same plane there is one common perpendicular, and only one.
36. The bases of a cylinder are congruent.
37. If a plane intersects a sphere, the line of intersection is a circle.
38. The volumes of two tetrahedrons that have a trihedral angle of one equal to a trihedral angle of the other are to each other as the products of the three edges of these trihedral angles.
39. In any polyhedron the number of edges increased by two is equal to the number of vertices increased by the number of faces.
40. Two similar polyhedrons can be separated into the same number of tetrahedrons similar each to each and similarly placed.

41. The volumes of two similar tetrahedrons are to each other as the cubes of any two corresponding edges.
42. The volumes of two similar polyhedrons are to each other as the cubes of any two corresponding edges.
43. If three face angles of one trihedral angle are equal, respectively, to the three face angles of another the trihedral angles are either congruent or symmetric.
44. Two spherical triangles on the same sphere are either congruent or symmetric if (a) two sides and the included angle of one are equal to the corresponding parts of the other; (b) two angles and the included side of one are equal to the corresponding parts of the other; (c) they are mutually equilateral; (d) they are mutually equiangular.
45. The sum of any two face angles of a trihedral angle is greater than the third face angle.
46. The sum of the face angles of any convex polyhedral angle is less than four right angles.
47. Each side of a spherical triangle is less than the sum of the other two sides.
48. The sum of the sides of a spherical polygon is less than 360°.
49. The sum of the angles of a spherical triangle is greater than 180° and less than 540°.
50. There can not be more than five regular polyhedrons.
51. The locus of points equidistant (a) from two given points; (b) from two given planes which intersect.

IV. SUBSIDIARY PROPOSITIONS IN MENSURATION

52. The volume of a frustum of (a) a pyramid or (b) a cone.
53. The lateral area of a frutsum of (a) a pyramid or (b) a cone of revolution.
54. The volume of a prismoid (without formal proof).

ADVANCED ALGEBRA

1 *Prerequisites.* The work recommended under this caption will ordinarily best be given after the pupil has taken the work outlined for the first course in algebra, and a course in plane geometry. This will permit of a wide choice of interesting problem material and applications of algebra to the field of geometry.

a *Simple functions of one variable.* Numerous illustrations and problems involving linear, quadratic and other simple functions including formulas from science and from common life. More difficult problems in variation than those included in the earlier course.

The following may serve to indicate the nature of material appropriate for inclusion here:

Problems in variation such as those concerning wind pressures as dependent on wind-velocity, of fuel-consumption as dependent on velocity of travel, of strength of beams as dependent on dimensions of cross-sections afford opportunities for applying the equation $y=kx$.

b *Equation in one unknown.* Various methods for solving a quadratic equation (such as factoring, completing the square, use of formula) should be given, but the method by completing the square should be recognized as fundamental.

In connection with the treatment of the quadratic a very brief discussion of complex numbers should be included.

Treatment of radical equations should be restricted to the solution of equations not more complicated than:

$$2x-\sqrt{3x+4}=4$$

By means of the graphic solution of equations of degree higher than the second, the pupil's confidence in his ability to find an approximate solution of any equation may be developed. Thus the problem "Find the size of the square to be cut from each corner of a sheet of tin 8″ x 10″, in order to make an open box of volume 40 cu. in." leads to a cubic equation whose approximate solution may be found from the graph.

The solution of problems involving the idea of maximum and minimum may also be taken up in this connection.

c *Equations in two or three unknowns.* The algebraic solution of linear equations in two or three unknowns and the graphic solution of linear equations in two unknowns should be given. The graphic and algebraic solution of a linear and a quadratic equation and of two quadratics that contain no first degree term and no xy term should be included.

d *Exponents, radicals and logarithms.* The definitions of negative, zero and fractional exponents should be given and it should be made clear that these definitions must be adopted in order to make such exponents conform to the laws for positive integral exponents. Reduction of radical expressions to those involving fractional exponents should be given as well as the inverse transformation. The fundamental operations on expressions involving radicals, and such transformation as:

$$\sqrt[n]{a/b}=\frac{1}{b}\sqrt[n]{ab^{n-1}},\quad \sqrt[n]{a\,b}=a\sqrt[n]{b},\quad \frac{a}{\sqrt{b}+\sqrt{c}}=\frac{a(\sqrt{b}-\sqrt{c})}{b-c}$$

should be included.

In this connection, care should be taken to make clear the advantage gained by such transformations. To "rationalize the denominator" without knowing why the operation is performed, is profitless labor. In close connection with the work on exponents and radicals there should be given as much of the theory of logarithms as is needed for their application to computation and sufficient practice in their use in computation to impart a fair degree of facility.

The graph of the equation $y=10^n$, with n taking value from 0 to 1, offers a good approach to the treatment of logarithms. Let n be successively 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{3}{8}$, $\frac{5}{8}$, $\frac{3}{4}$, $\frac{7}{8}$, and compute the corresponding value of *y*, by taking the square root of ten, the square root of this result, et cetera. From the graph the pupil constructs, embodying these results, he can read the number corresponding to any value of *n*, and the value of n corresponding to any number, thus enabling him to perform multiplications and divisions by means of the graph. The printed tables should then be taken up.

e *Arithmetic and Geometric Progressions.* The formulas for the nth term and the sum of n terms should be derived and applied to significant problems, such as computing the value of savings deposited regularly, and the payment of debts in equal periodical sums, as in purchasing a home on monthly installments.

f *Binomial Theorem.* A proof for positive integral exponents should be given and it should be stated that the formula holds for negative and fractional exponents under suitable restrictions, but proofs for these cases should be omitted. The problems may include the use of the formula for these exponents as well as positive integral exponents: as, for instance, in the derivation of the approximation formula:

$$\sqrt{a^2 \pm b} = (a^2 \pm b)^{\frac{1}{2}} = a \pm \frac{b}{2a}$$

TRIGONOMETRY

Introductory work. The course in trigonometry presupposes that the pupil has had work such as is indicated in the section on numerical trigonometry, on page 14. If such work has not already entered into the pupil's mathematics course, it should be given as the introduction to the present course, with a time-allotment of about six weeks.

In particular, the pupil should learn the definitions of sine, cosine and tangent as pure numbers, depending only on the size of the angle. His first contact with these ratios will be made through measurement and tabulation from his own drawings of right triangles. The pupil should make his own table for a few angles by

graphical construction and computation to two places of decimals, before using the printed tables. He will learn to find any function when one is known, by drawing a right triangle, marking two sides to correspond to the numerator and denominator of the given function, computing the remaining side by the pythagorean theorem, and then reading off the new desired function directly from the figure.

Problems and measurements. The solution of problems in indirect measurement of heights and distances, based on measurements actually made by the pupils, will make them familiar with the complete solution of a right triangle by the aid of a table of sines, cosines and tangents when any two parts (including at least one side and excluding the right angle) are given. If possible, the transit should be used to carry out some of the simpler operations of surveying. When no transit is available, apparatus should be improvised and used for measuring angles.

Diagrams. Every problem should be accompanied by a sketch or diagram to ensure that the pupil understands the meaning of each step of the work. An accurate graphical solution may be used as a check on the correctness of the numerical computation.

Neatness. Neatness and systematic arrangement of work should be insisted upon from the beginning of the course.

The course in trigonometry will be continued beyond the introductory work above referred to, by considering the following topics:

A *Angles in general, and functions of any angle.* Heretofore the angles considered have been largely acute angles, and the functions considered have been defined only for acute angles. It will be necessary to extend the idea of angle to cover the idea of angular magnitude in general, and to extend the idea of sine, cosine, and tangent to apply to angles other than those in the first quadrant. In this connection, it becomes necessary to fix clearly the algebraic signs of the various functions. For this purpose, the line representations of the functions in the unit-circle are probably most efficient.

B *Use of tables: reduction to the first quadrant.* The tables give values of the functions in the first quadrant only. It is necessary for the pupil now to become able to find the sine, cosine and tangent of angle in any quadrant: that is, to derive equations of the type $\cos(90 + A) = -\sin A$, and for this purpose, again, the line-representations in the unit circle are probably most useful.

C *Theory and use of logarithms.* The work with logarithms may wisely minimize the theory. Four-place tables should be used. The slide rule is recommended for use in computation and checking.

D Trigonometric Equations. The solution of trigonometric equations should be restricted to those of about the order of difficulty of:

$$3 \sin x - 2 \cos x = 2; \text{ and } \tan (x - 45) = \text{ctn } x$$

E Formulas needed for the solution of problems. The pupil should be made to see that all oblique triangles can be solved by dissecting them into right triangles, and solving the latter in detail by applying the methods already learned. He should be led to see that the cumbersome and tedious solutions that are made necessary by this procedure can be replaced by more elegant and expeditious solutions when certain formulas, such as the "law of sines" and the "law of cosines," are available.

By leading the pupil to see the advantages to be gained by the use of logarithms in the solution of problems he will appreciate the value of such formulas as $\tan \frac{A}{2} = \frac{r}{s-a}$

Throughout the work in trigonometry, stress should be laid upon problem-solving and work on formulas should be kept from degenerating into purposeless juggling with symbols, by making clear the need for each group of formulas derived, and the advantage to be gained by their use.

The teacher will easily find numerous applications of trigonometric formulas to engineering, navigation, and surveying. Problems secured from architects and engineering designers, involving applications of trigonometry, are enormously more valuable for class use than mere text-book problems. Pupils should be encouraged to seek such problems for class use.

FORMULAS

A The following formulas should become part of the pupil's mental equipment, on a par with the multiplication tables:

I $\tan x = \frac{\sin x}{\cos x}$; $\text{ctn } x = \frac{\cos x}{\sin x}$;

$\sec x = \frac{1}{\cos x}$; $\csc x = \frac{1}{\sin x}$

II *The Pythagorean formulas:*

$\sin^2 x + \cos^2 x = 1$

$\tan^2 x + 1 = \sec^2 x$

$1 + \text{ctn}^2 x = \csc^2 x$

III *The Addition formulas:*

$$\sin(x+y)=\sin x\cos y+\cos x\sin y$$
$$\cos(x+y)=\cos x\cos y-\sin x\sin y$$
$$\tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\tan y}$$

IV *The Double-angle formulas:*

$$\sin 2x=2\sin x\cos x$$
$$\cos 2x=\cos^2 x-\sin^2 x$$
$$=2\cos^2 x-1$$
$$=1-2\sin^2 x$$
$$\tan 2x=\frac{2\tan x}{1-\tan^2 x}$$

V *a The Law of Sines:*

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$

b The Law of Cosines:

$$a^2=b^2+c^2-2bc\cos A.$$

B The following sets of formulas are very useful, but not sufficiently important to justify the requirement that they be memorized:

VI *Formulas read off from a figure, as:*

$\cos(90°+x)$, $\sin(-x)$, *et cetera, in terms of functions of* x.

VII *The Half-Angle Formulas:*

$$\sin y=2\sin\frac{y}{2}\cos\frac{y}{2}$$
$$1+\cos y=2\cos^2\frac{y}{2}$$
$$1-\cos y=2\sin^2\frac{y}{2}$$
$$\sin\frac{y}{2}=\pm\sqrt{\frac{1-\cos y}{2}}$$
$$\cos\frac{y}{2}=\pm\sqrt{\frac{1+\cos y}{2}}$$
$$\tan\frac{y}{2}=\pm\sqrt{\frac{1-\cos y}{1+\cos y}}$$

VIII *Special formulas for solving triangles:*

a *Law of tangents:*

$$\frac{a-b}{a+b}=\frac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(A+B)}$$

b *When three sides are known:*

$$\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

where $s=\frac{1}{2}(a+b+c)$

TWELFTH YEAR MATHEMATICS

Opportunity to continue the study of mathematics beyond trigonometry should be presented by providing courses such as the following:

Elementary Statistics. Meaning and the use of fundamental concepts, simple frequency distributions with graphic representations of various kinds, measure of central tendency.

Mathematics of Finance. Interest, annuities, sinking funds, depreciation, the mathematical theory of insurance, the mathematics of building and loan associations.

Shop Mathematics. Graphical representation, logarithmic computation, the use of tables of various kinds, the slide rule, empirical formulas. (This course may profitably replace the conventional eleventh year course for some groups of pupils.)

Descriptive Geometry. Involving coöperation between the drawing-room and mathematical teachers; stress laid at first upon principles, rather than on highly finished plates.

General Mathematics. A course in composite mathematics, involving the fundamental topics of college algebra, trigonometry, analytic geometry and calculus. A number of good texts are available. To be recommended for such schools as have a strong teacher and a good group of twelfth-year pupils.

Elementary Calculus. The general notion of a derivative as a limit; application of derivatives to easy problems in rates and in maxima and minima; simple cases of inverse problems, as finding distances from velocities; approximate methods of summation leading to the notion of integration; application to simple cases of motion, area, volume and pressure.

Not all of these subjects can be offered in all schools, nor would it be desirable to do so. While all of the subjects listed here can be studied profitably by pupils who have done the work outlined through the eleventh year, the pupil's vocational or later educational needs,

as well as the interests of the pupils who "like" mathematics, must be considered in determining the courses to be offered in any particular school.

BIBLIOGRAPHY

High school mathematics teachers should be familiar with as many of the following publications as possible:

1 The Reorganization of the First Courses in Secondary School Mathematics: Secondary School Circular No. 5, Bureau of Education, Washington, D. C. 1920.

2 Junior High School Mathematics: Secondary School Circular, No. 6, Bureau of Education, Washington, D. C. 1920.

3 Cardinal Principles of Secondary Education, Bulletin 1920, No. 1, Bureau of Education, Washington, D. C.

4 The Problem of Mathematics in Secondary Education, Bulletin 1920, No. 1, Bureau of Education, Washington, D. C.

g Scientific Method in the Reconstruction of Ninth Grade Mathematics, H. O. Rugg and J. R. Clark: University of Chicago Press, 1918.

6 Psychology of High School Subjects: C. H. Judd, Ginn & Co., 1915.

7 The Teaching of Algebra (including Trigonometry) Nunn: Longmans, 1914.

8 The Teaching of Geometry: D. E. Smith: Ginn, 1911.

9 The Teaching of Mathematics in Secondary Schools, Schultze, Macmillan, 1916.

10 Fundamental Concepts of Algebra and Geometry: J. W. Young, Macmillan, 1911.

11 How We Think: John Dewey: D. C. Heath & Co., 1910.

12 Introduction to Mathematical Literature: G. A. Miller.

13 The Human Worth of Rigorous Thinking: Keyser: Columbia University Press, 1916.

14 Mathematical Education: Carson: Ginn & Co., 1913.

15 A Study of Mathematical Education: Benchara Branford: Clarendon Press, 1908.

16 First Year Algebra Scales: H. G. Hotz: Teachers College Contributions to Education, No. 190, 1918.

17 Experimental Tests of Mathematical Ability and Their Prognostic Value, Agnes L. Rogers: Teachers College Contributions to Education, No. 89, 1918.

JOURNALS

High school mathematics teachers should be familiar with the following monthly publications:

1 Mathematics Teacher: devoted to the interests of teachers of mathematics in the junior and senior high school.

2 The School Review: devoted to the interests of the high school in general.

3 The Mathematical Monthly: devoted primarily to the interests of the college teacher of mathematics, and having much material of interest to high school teachers.

4. School Science and Mathematics: a journal for science and mathematics teachers in the high school.